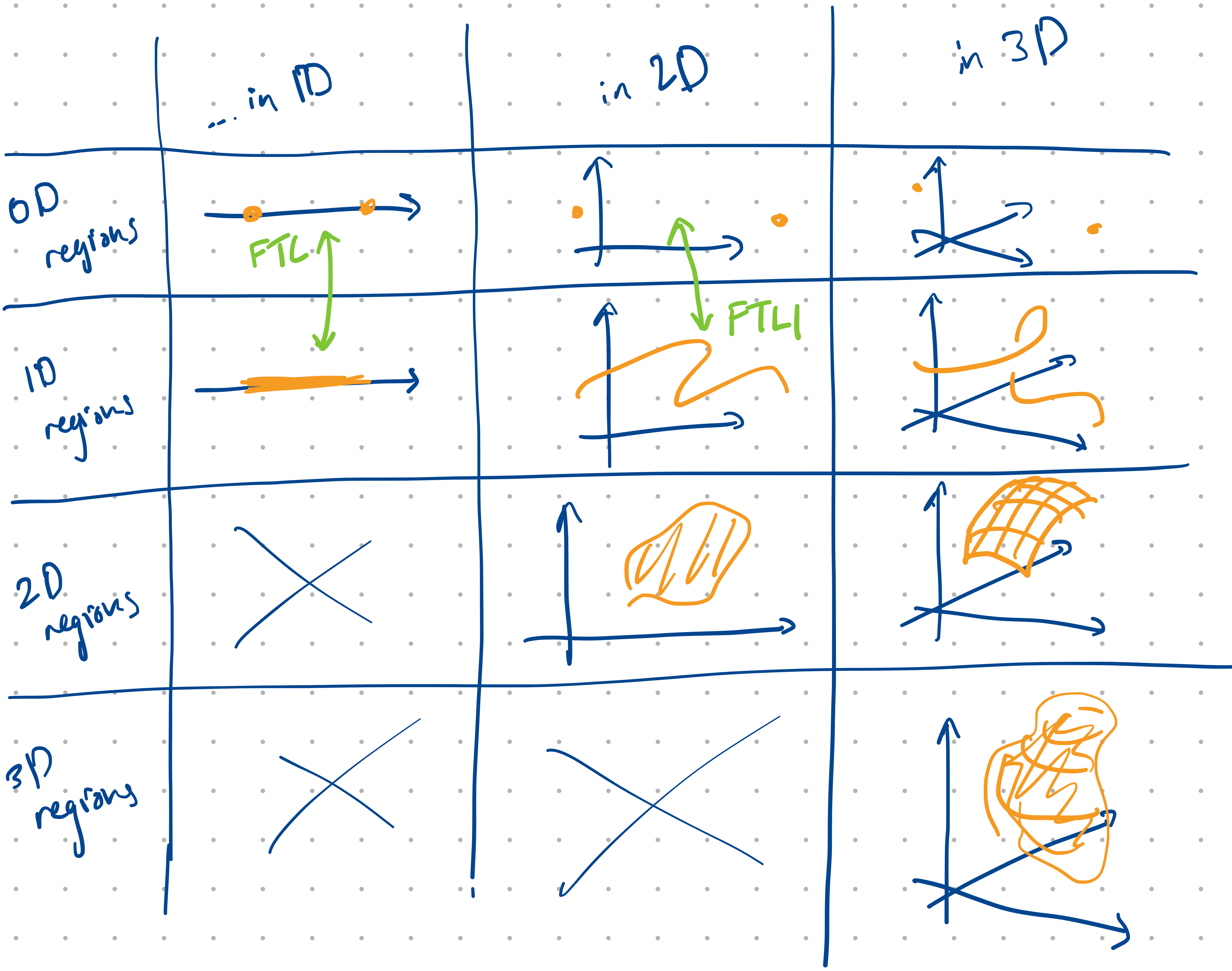


⚠ FTLI, "path-independence" etc...

are about <u>vector fields</u>, not scalar fields ($\mathbb{R}^2 \longrightarrow \mathbb{R}$)

i.e. they are about problems like

$$\int_C \vec{F} \cdot d\vec{r} \qquad \underline{\underline{\text{NOT}}} \qquad \int_C f \, ds$$

SVC: scalar functions e.g. $f(x)=x^2$ $\xrightarrow{\text{differentiate}}$ scalar functions $f'(x)=2x$ ①

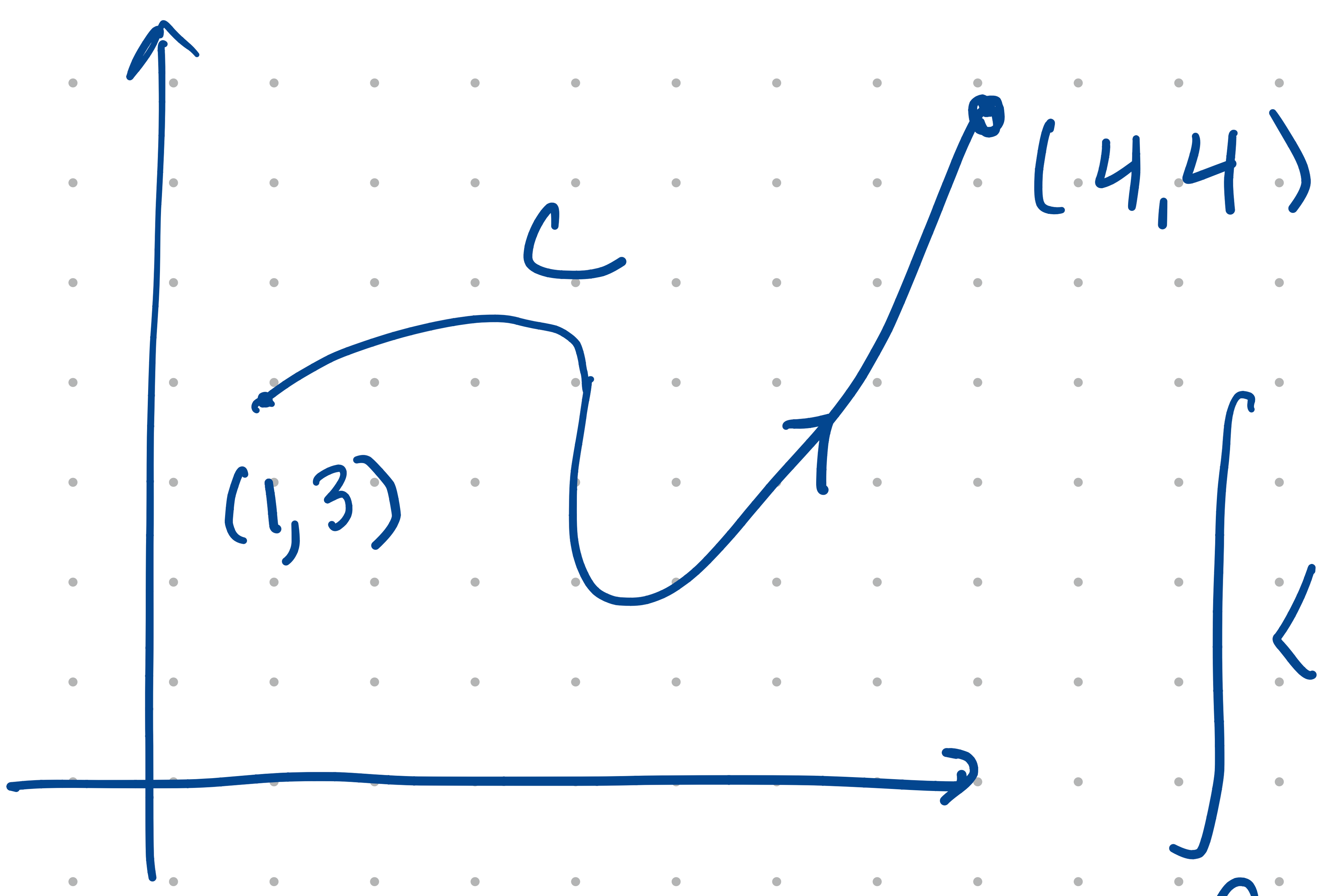
MVC: scalar functions $f(x,y)=x^2+y^2$ $\xrightarrow[\nabla]{\text{differentiate}}$ vector field $\nabla f(x,y)=\langle 2x, 2y\rangle$ ②

⚠ Essential difference: Everything (reasonable) in ① comes from something on the left, but that's not the case in ②.

e.g. $\langle \underset{P}{x^2}, \underset{Q}{y+x^2} \rangle$.

$P_y \neq Q_x$ so this is not a gradient.

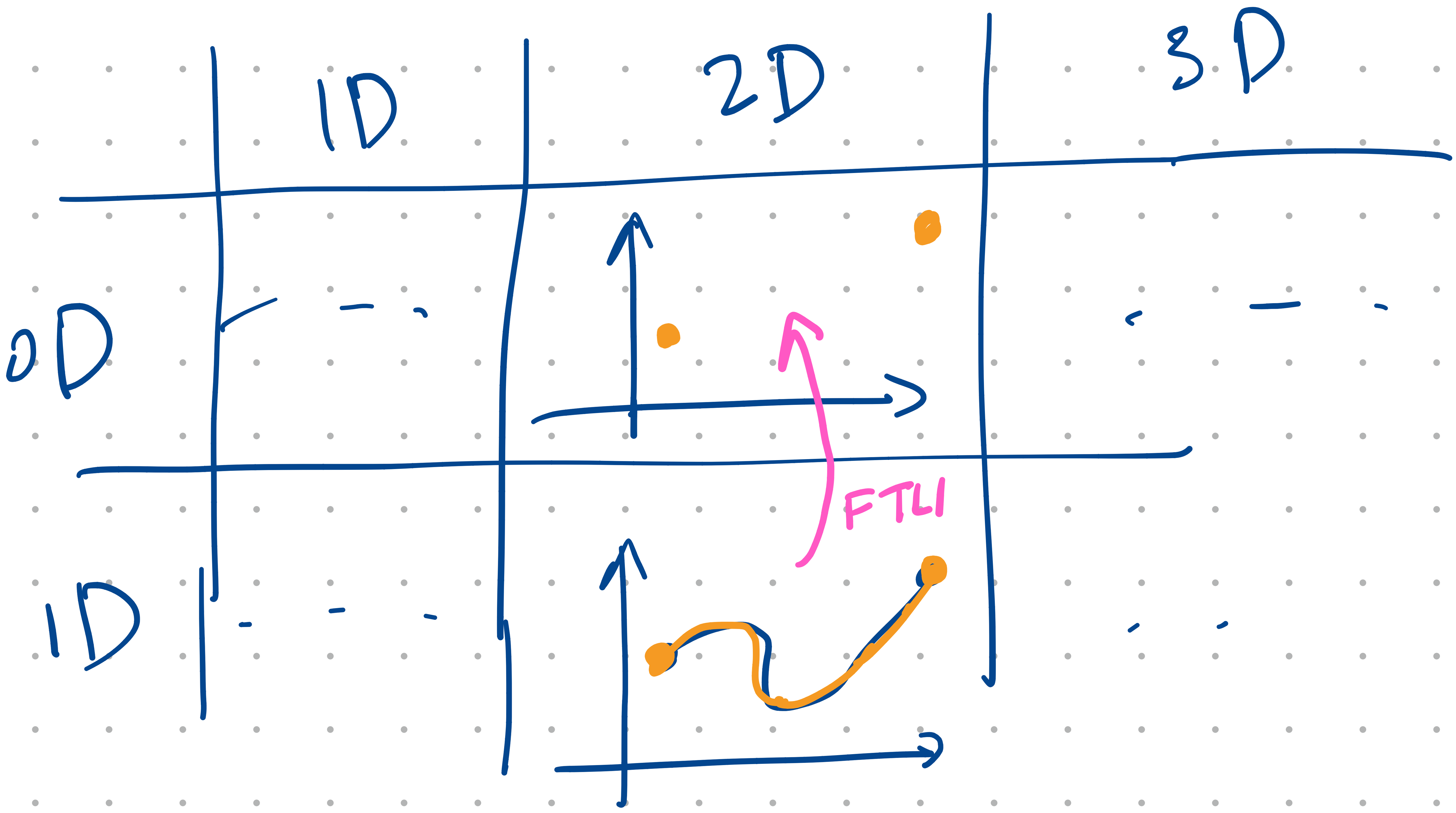
ex)



$$\int_C \langle 2x, 2y \rangle \cdot d\vec{r}$$

$$= \int_C \nabla (x^2+y^2) \cdot d\vec{r}$$

$$= (x^2+y^2) \Big|_{(x,y)=(1,3)}^{(4,4)} = 32 - 10 = \boxed{22}$$



$\vec{F}$ a vector field defined on some region R.

②

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$ for all paths C_1, C_2 in R that have the same endpoints

"path-independence"

①

$$\oint_C \vec{F}\cdot d\vec{r} = 0$$

along every closed loop C in R

"conservative"

③

there exists a "potential function" f s.t. $\nabla f = \vec{F}$.

Clairaut's Thm.

if R is simply connected.

④

$\nabla \times \vec{F} = \vec{0}$ (in 2D, $P_y = Q_x$)

To show $\vec{F}$ is conservative:

③ could try to find a potential function.

④ if R is simply connected, then can use $P_y = Q_x$ to check.

To show $\vec{F}$ is not conservative:

③ show that integration results in a contradiction.

④ show $P_y \neq Q_x$ (recommended)

①, ② Find a loop C such that $\oint_C \vec{F} \cdot d\vec{r} \neq 0$.

1) Look @ $(0,-5)$

$$\leftarrow \qquad \rightarrow \qquad \downarrow \qquad \uparrow$$

$\langle x, -y \rangle$ @ $(0,-5)$ is $\langle 0, 5 \rangle$

2) $\mathbb{R}^2$ is simply conn, and $P_y = 0 = Q_x$ so it is conservative.

Try to find $f(x,y)$ s.t. $\nabla f(x,y) = \langle x, -y \rangle$.

$$f_x(x,y) = x \quad \text{so} \quad f(x,y) = \frac{1}{2}x^2 + \underbrace{C(y)}_{\text{function of } y}$$

$$-y = f_y(x,y) = 0 + C'(y)$$

thus $C(y) = -\frac{1}{2}y^2 + D$ ← constant.

e.g. $f(x,y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 + 42$.

3) @ $(0,-5)$ have $\longleftarrow$ so A. $\langle y,-x\rangle$

4) $P_y = 1 \neq -1 = Q_x$

so it is not conservative.

Alternative:

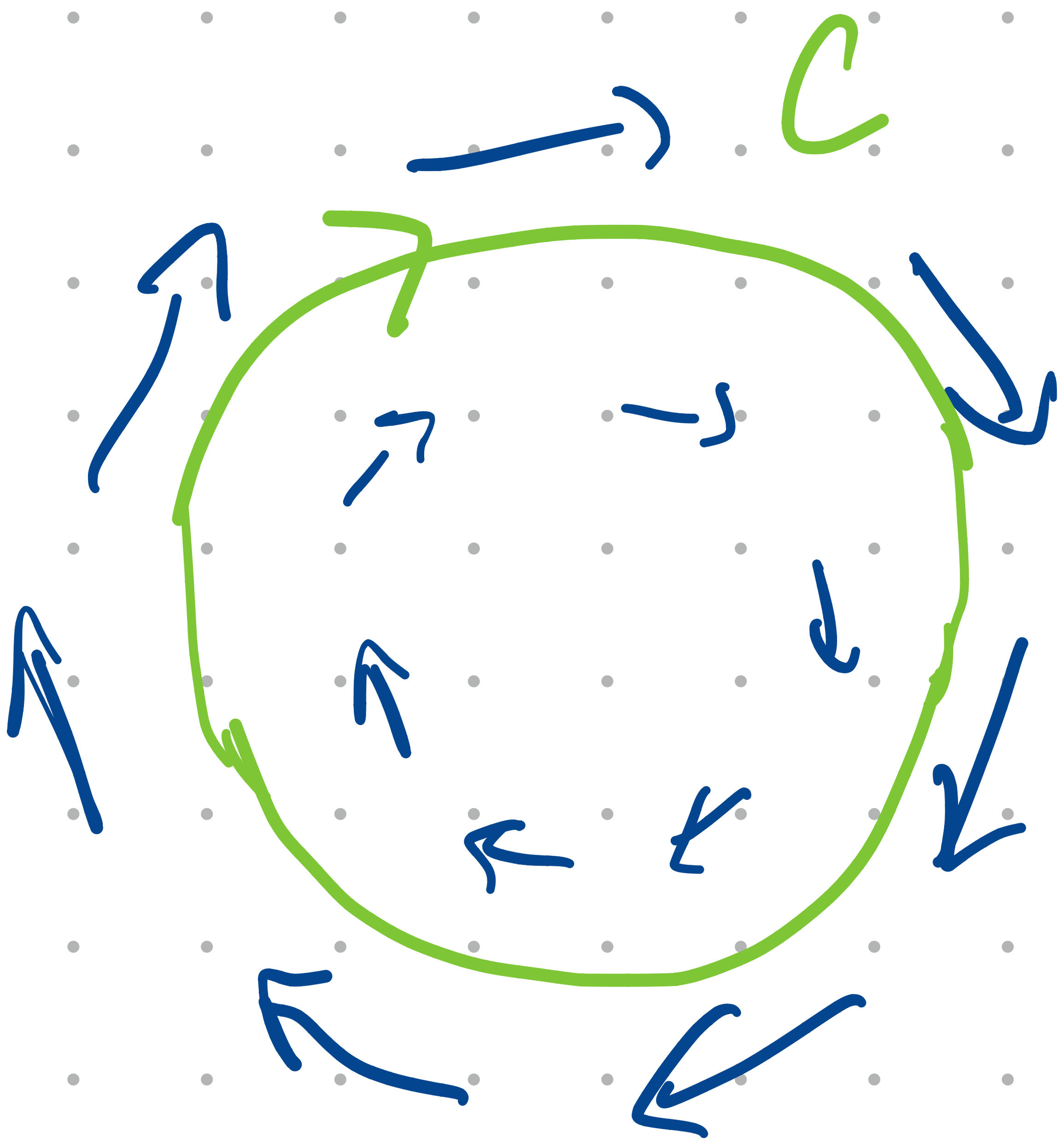
$f_x(x,y) = y$ so $f(x,y) = xy + C(y)$

$$-x = f_y(x,y) = x + C'(y)$$

impossible to have $C'(y) = -2x$.

no x here!!

Alternative:

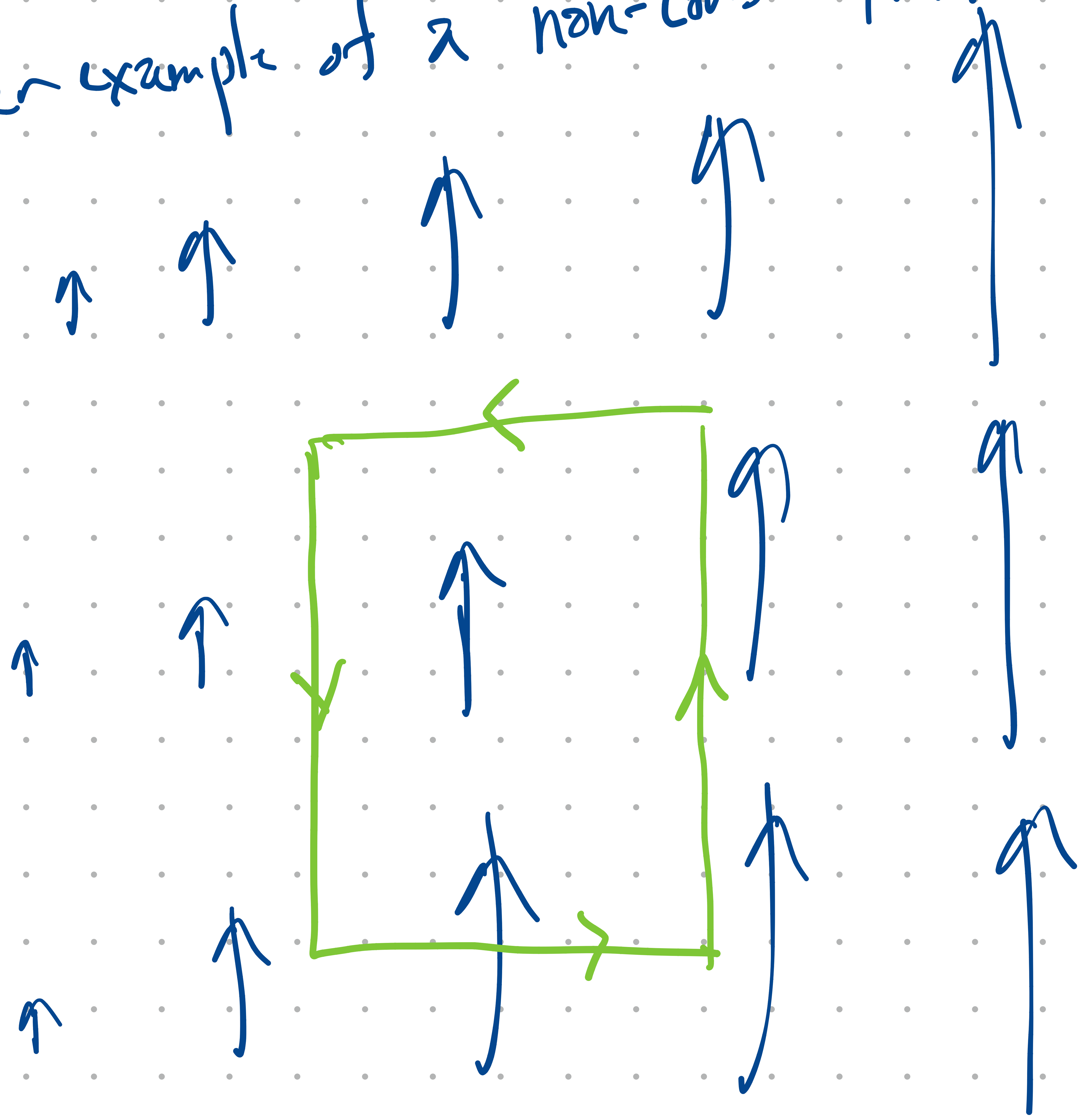


$C: \quad \vec{r}(t) = \langle \sin t, \cos t \rangle \quad 0 \le t \le 2\pi$

$$\int_C \langle y, -x \rangle \cdot d\vec{r}$$

$$= \int_0^{2\pi} \left((\cos t)^2 + (\sin t)^2 \right) dt = 2\pi \neq 0.$$

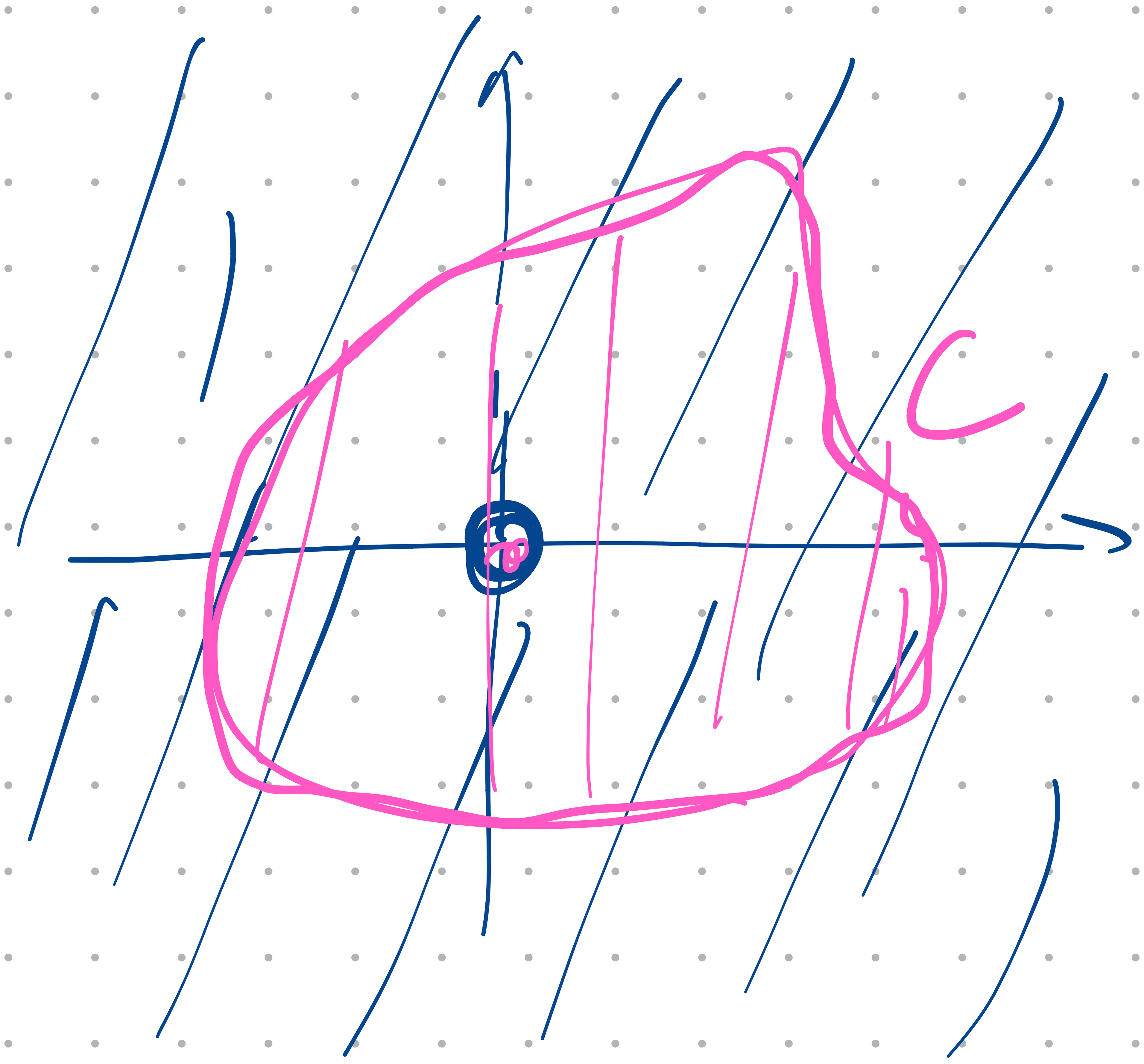
Another example of a non-cons. field.



5) In this case, $Q_x = P_y$.

However:

Domain of $\vec{F}$ is $\mathbb{R}^2 - 0$.

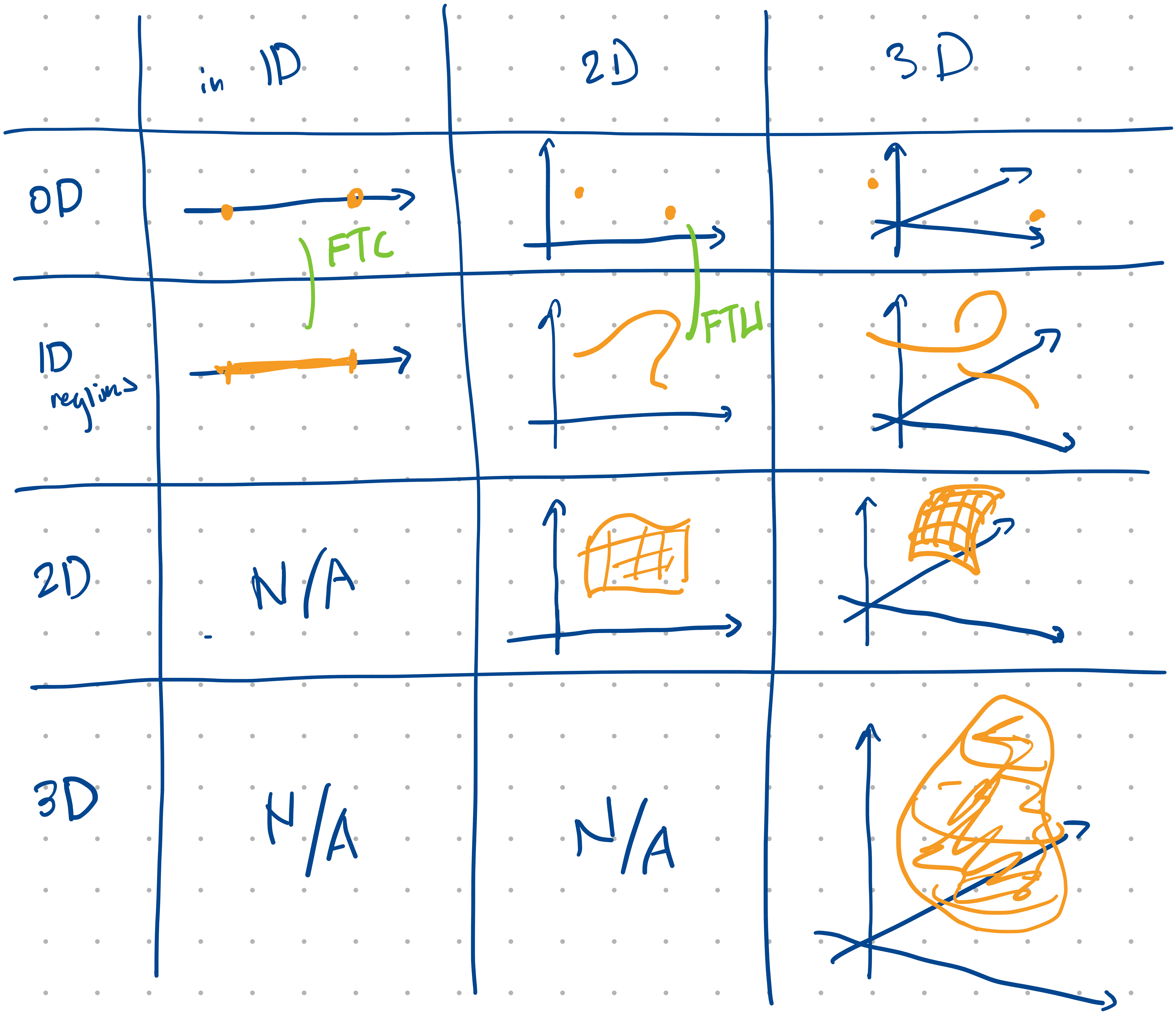


<u>Defn</u>. R (region) is simply connected if every loop in R encloses points only in R.

e.g. C is a loop in $\mathbb{R}^2 - (0,0)$ but it encloses $(0,0)$ which is not in $\mathbb{R}^2 - (0,0)$, so that region is <u>not simply connected</u>.

Exercise: Find a loop C s.t. $\oint_C \vec{F} \cdot d\vec{r} \neq 0$.

for the vec. field in 5.



⚠ FTLI, "path-indep" etc. are about <u>vector fields</u>.

i.e. they are about

$$\int_C \vec{F}\cdot d\vec{r} \qquad \underline{\text{NOT}} \qquad \int_C f \, ds$$

SVC: scalar fn. $\xrightarrow{\text{diff.}}$ ① scalar function

e.g. $f(x) = x^2$ $\qquad$ $f'(x) = 2x$.

MVC: scalar fn $\xrightarrow[\nabla]{\text{diff}}$ ② vector field

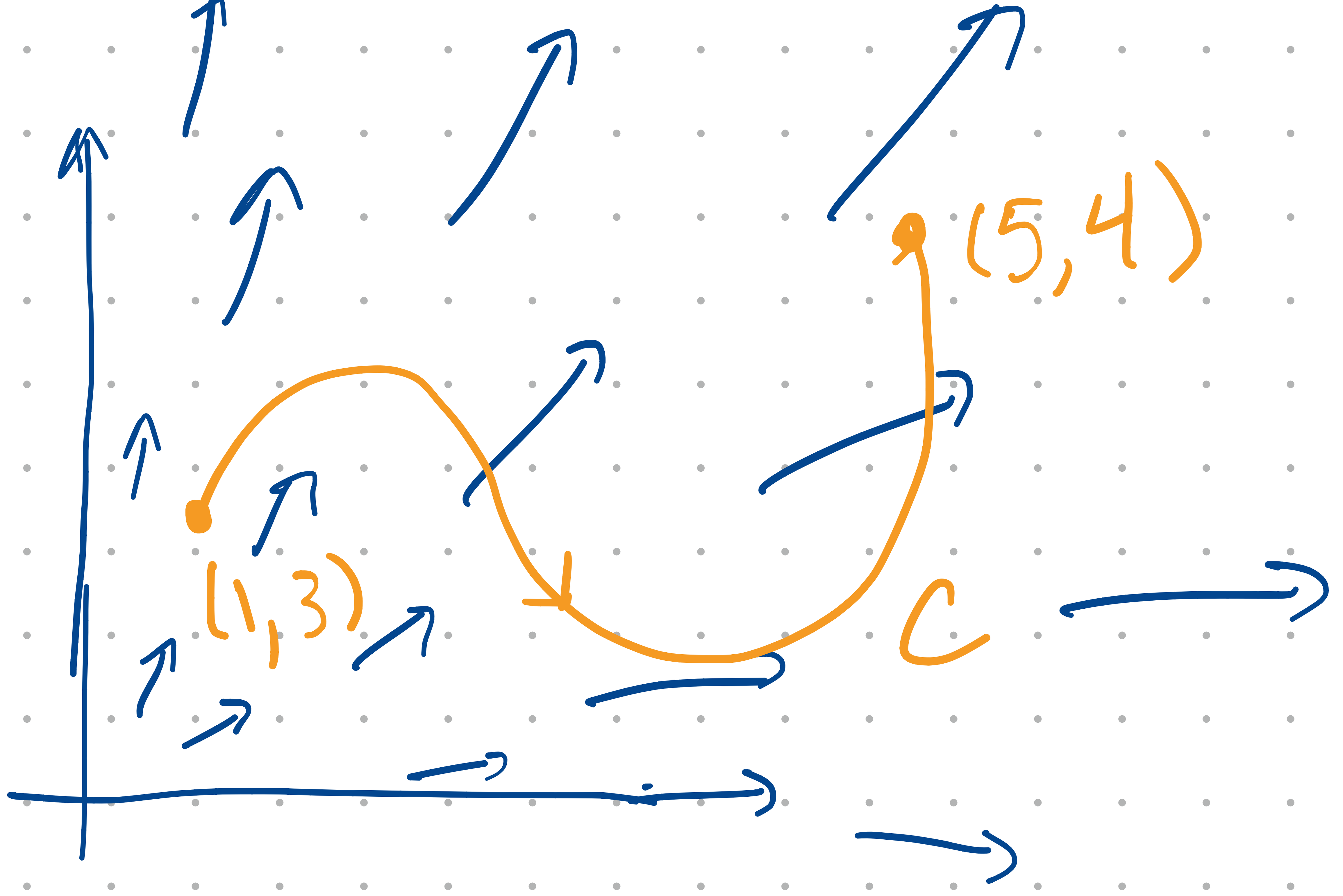
e.g. $f(x,y) = x^2+y^2$ $\qquad$ $\nabla f(x,y) = \langle 2x, 2y\rangle$

⚠ In SVC: every reasonable scalar fn ① is a derivative

But in MVC: not all vector fields ② are gradients.

e.g. $\langle x^2, x^2+y^2\rangle$ is not a gradient vector field.

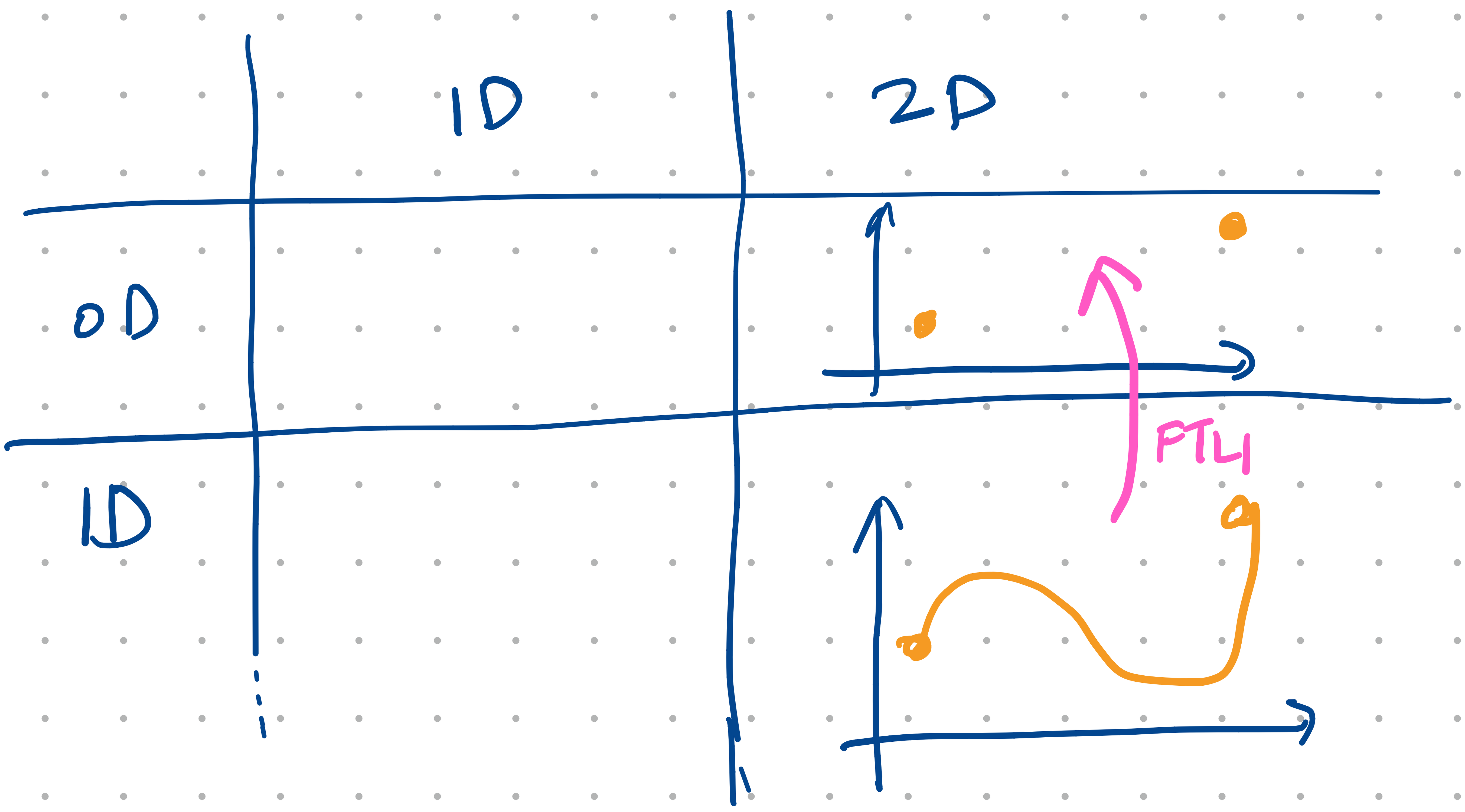
(why?)



$$\vec{F} = \langle 2x, 2y \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla(x^2+y^2) \cdot d\vec{r}$$

$$\overset{\text{FTLI}}{=} (x^2+y^2)\Big|_{(x,y)=(1,3)}^{(5,4)} = 41 - 10 = 31$$



$\vec{F}$ a vec. field defined on region R.

① $\oint_C \vec{F}\cdot d\vec{r} = 0$ for every closed loop in R

$\Longleftrightarrow$

② "path-independence" $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$ for every pair of paths C_1, C_2 in R that share the same endpoints

$\vec{F}$ is conservative (on R)

③ there exists a function f defined on R s.t. $\nabla f = \vec{F}$. ("potential")

Clairaut's

if R is "simply connected."

④ $\nabla \times \vec{F} = \vec{0}$ (in 2D this means $P_y = Q_x$ where $\vec{F} = \langle P, Q \rangle$)

How to show $\vec{F}$ is conservative?

③ Actually find the potential function.

④ If R is simply connected, then check $P_y = Q_x$.

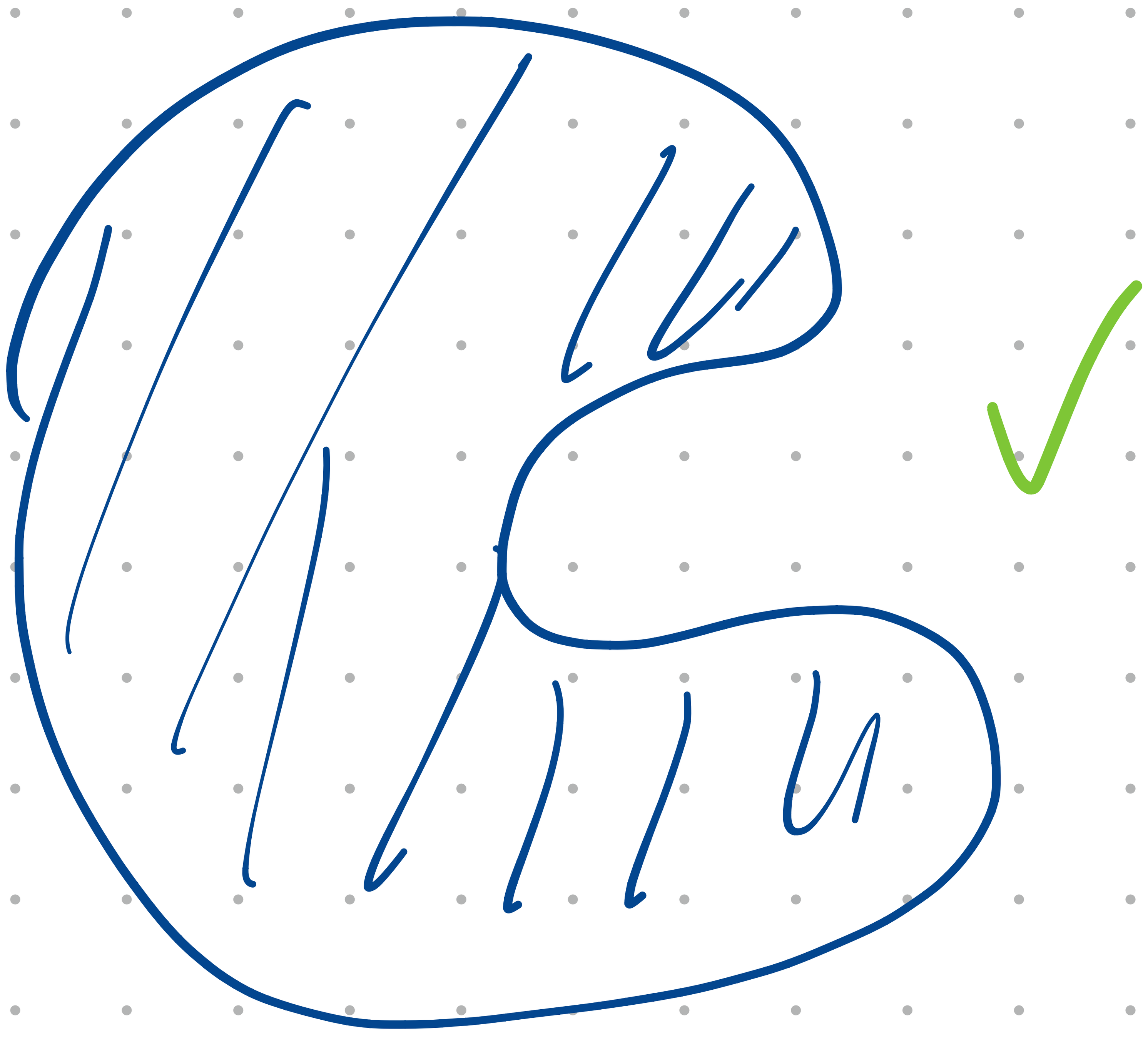
How to show $\vec{F}$ is not conservative?

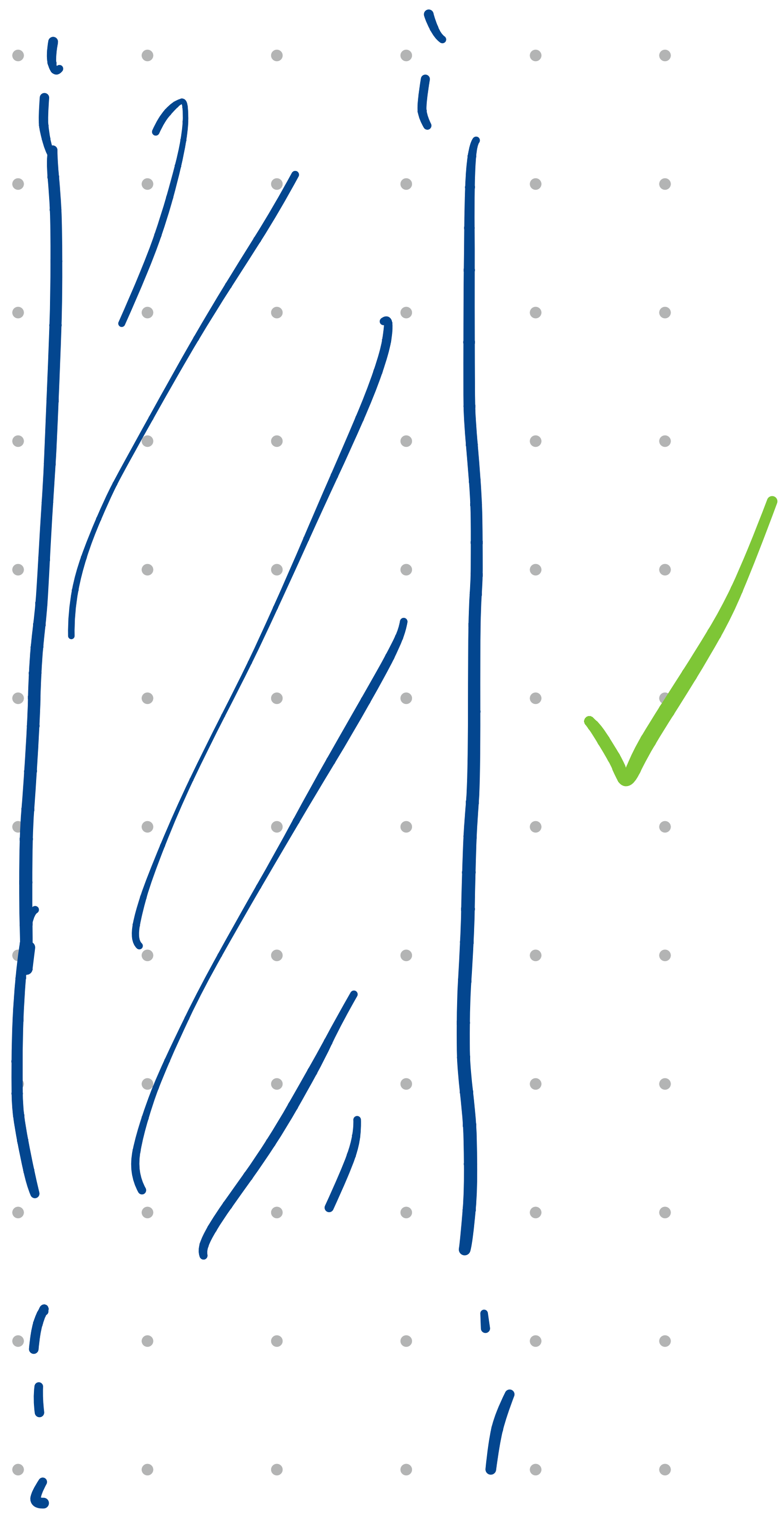
①,② find some loop C s.t. $\oint_C \vec{F} \cdot d\vec{r} \neq 0$.

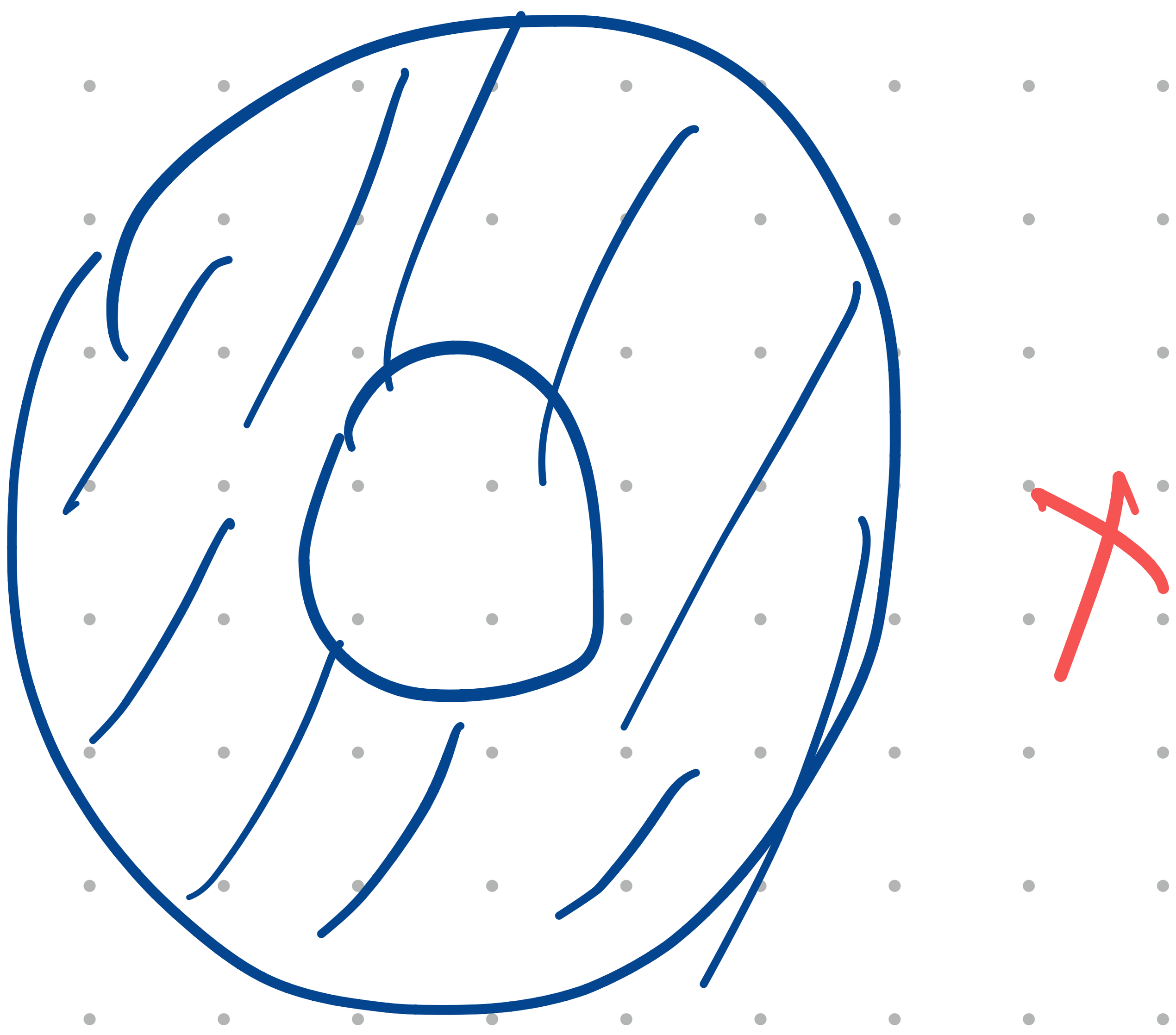
③ Try to integrate and show it results in a contradiction.

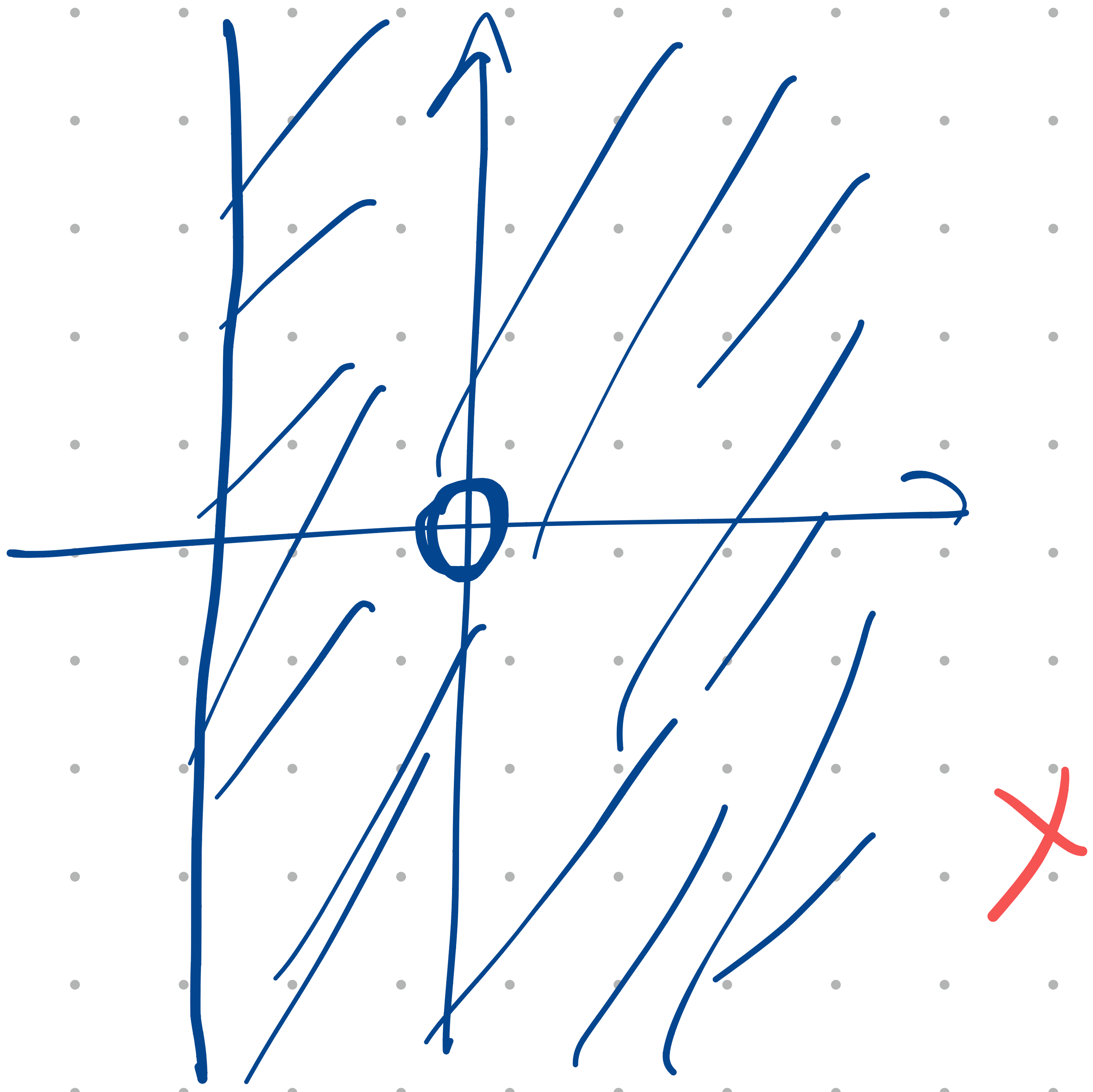
④ show $P_y \neq Q_x$ (recommended)

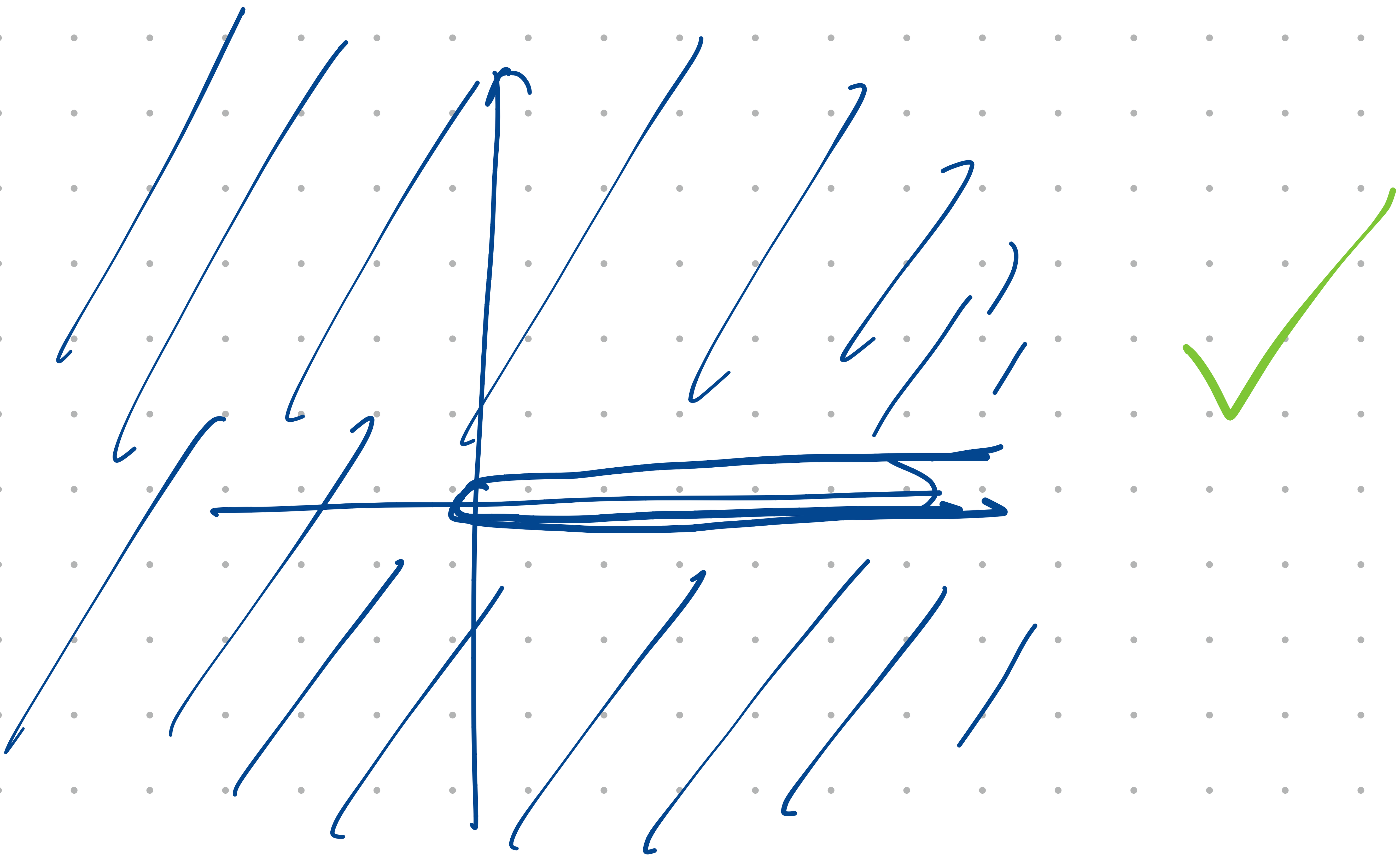
A region in 2D is simply connected if it has no holes. Equivalently, every loop in the region enclosed only points belonging to the region.











1) @ $(0,-5)$

$\leftarrow \qquad \rightarrow \qquad \downarrow \qquad \uparrow$

$\langle x,-y\rangle$ @ $(0,-5)$ is $\langle 0,5\rangle$

2) domain is $\mathbb{R}^2$, which is simply conn.

$P_y = 0 = Q_x$ ✓ so it's conservative.

$$f_x(x,y) = x \Longrightarrow f(x,y) = \frac{1}{2}x^2 + \underbrace{C(y)}_{\text{function of } y}$$

$$\Downarrow$$

$$-y = f_y(x,y) = 0 + C'(y)$$

thus $C(y) = -\frac{1}{2}y^2 + D \leftarrow$ const

so an example of a potential fn is

$$f(x,y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 + 42$$

3) $\langle y, -x \rangle$ @ $(0,-5)$ is $\langle -5, 0 \rangle$

i.e. $\leftarrow$

so first picture.

4) It's not conservative:

$$P_y = 1 \neq -1 = Q_x$$

<u>Alternative:</u>

Assume $\nabla f = \langle y, -x \rangle$.

$$f_x(x,y) = y \implies f(x,y) = xy + \underbrace{C(y)}_{\text{fn. of } y}$$

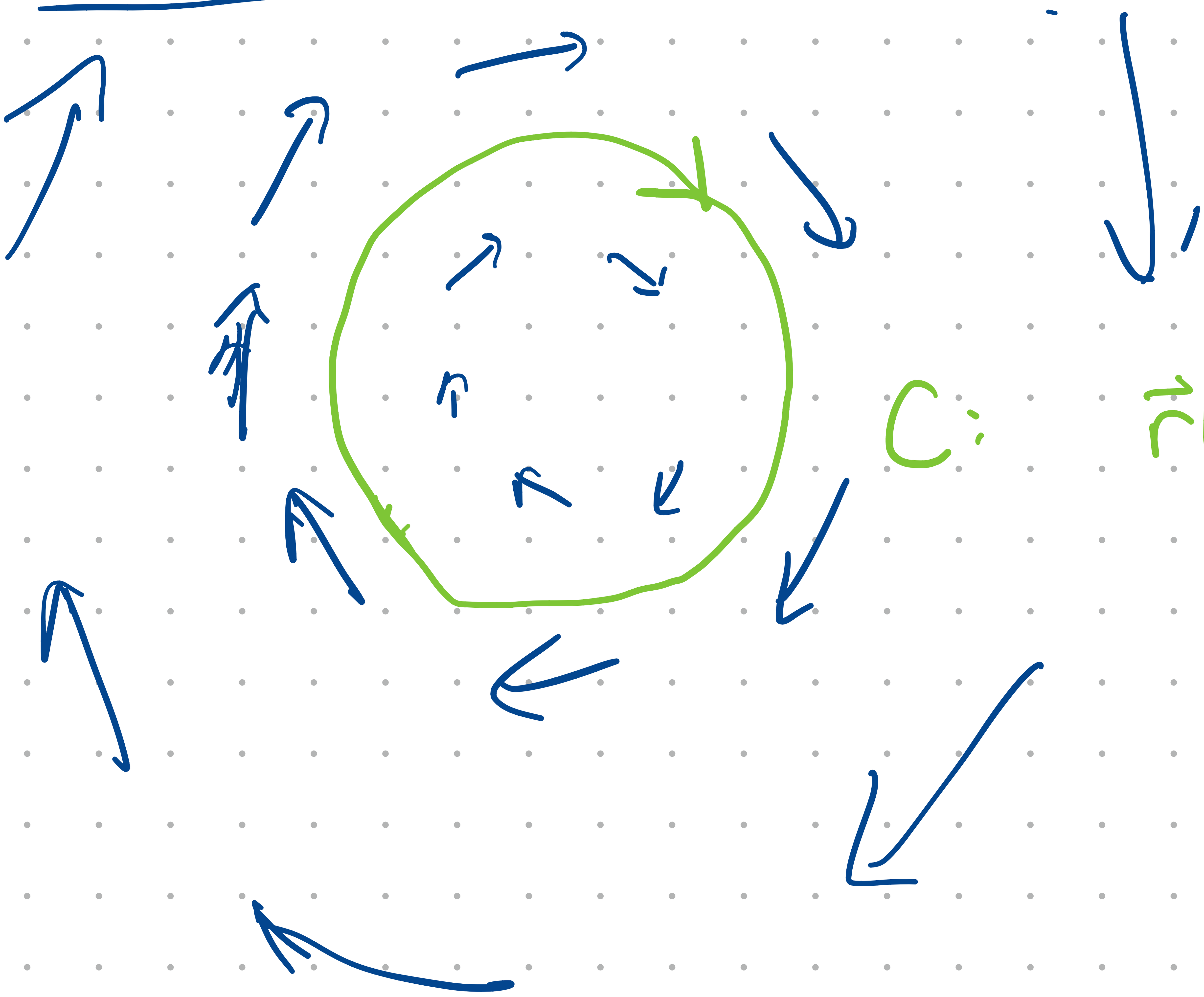
$$\Downarrow$$

$$-x = f_y(x,y) = x + C'(y)$$

$$C'(y) = -2x \quad \text{impossible}$$

involves y only.

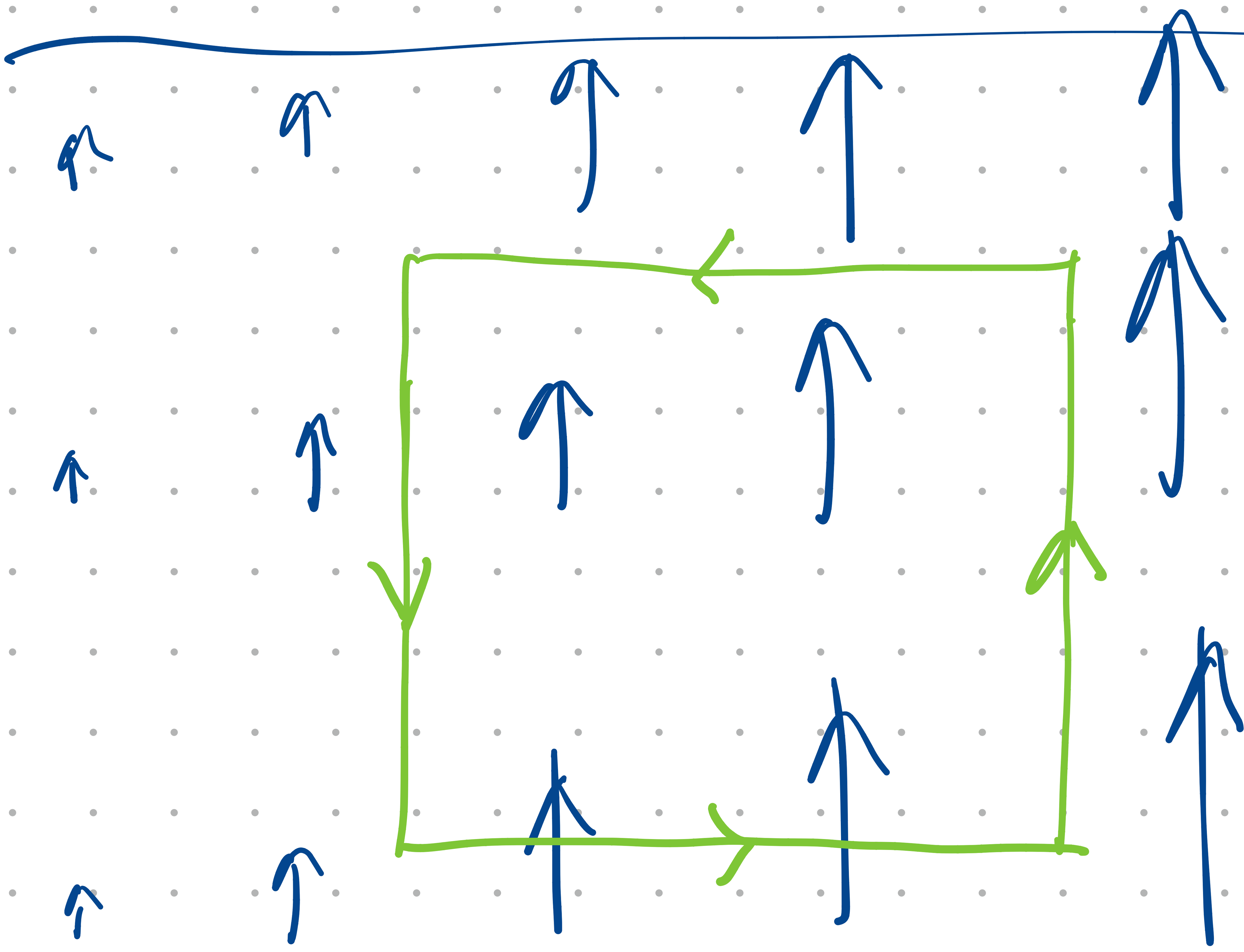
Alternative:



C: $\vec{r}(t) = \langle \sin t, \cos t \rangle$

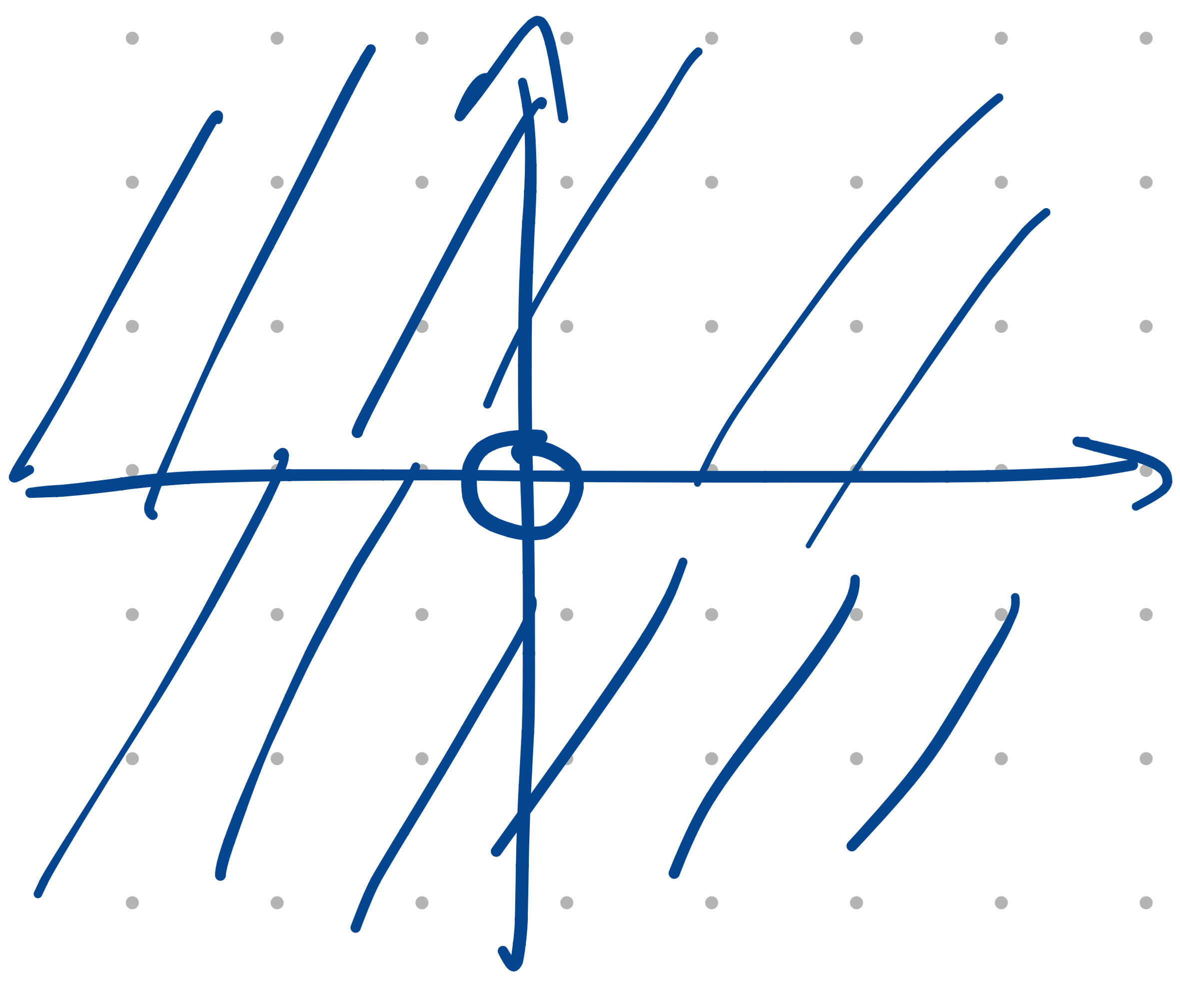
$0 \le t \le 2\pi$

Exercise: $\displaystyle\int_C \langle y, -x \rangle \cdot d\vec{r} = 2\pi \neq 0.$



positive integral so this vec. field is not conservative.

5) $P_y = Q_x$ (check!) however

the domain is  $\mathbb{R}^2 - (0,0)$

and this is not simply connected.

In fact $\vec{F}$ is not conservative.

Exercise: $$\oint_C \vec{F} \cdot d\vec{r} \neq 0$$

a loop around the origin